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Flow Measurement Uncertainty Assessment

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1 Abstract

Flow is a derived measurement that is invariably obtained from a number of measurements with a functional relationship that must be taken into account when analysing the measurement uncertainty. These relationships are often complex and frequently the results are input into a larger measurement system for the allocation of oil and gas production and sales.

Assessing of the uncertainty of a Custody flow measurement is not a trivial undertaking encompassing the calibration of all instrumentation, calibration of the flow element, regular proving and verification, the flow algorithms and correction of quantities to base conditions. Where the measurement is an input to a system for allocation to producers the uncertainty must be further propagated through algorithms to find the Producers uncertainty.

This paper discusses the benefits of assessing uncertainty to ensure Custody flow measurements comply with the required standards; and, in understanding the impact of the allocation uncertainty on the Producers. This includes a brief overview of uncertainty exposure and the relevant International Standards.

Techniques to combine and propagate uncertainties to find the uncertainty of a flow measurement and of a product allocation are described with examples:

1. Quadrature or Root Sum Square (RSS) with sensitivity terms by partial differentiation
2. Perturbation with RSS with numerical deviation of inputs to find sensitivity
3. Monte Carlo Simulation (MCS) numerical combination and propagation of uncertainty

The techniques are each compared to demonstrate that each gives the same result for simple applications. The advantages and disadvantages of each approach are discussed with the most appropriate for particular applications including hybrid uncertainty modeling which can use all three approaches. Detection of bias in some functional relationships is examined along with “Method Uncertainties” inherent in some fluid property algorithms.

The uncertainty of a Custody flow measurement and the allocation to Producers can be effectively assessed with a number of independent techniques to combine and propagate uncertainties.

2 Introduction

A Custody Transfer point can be defined as; a location where a fluid is being measured to determine, the quantity and quality, of fluid transferred from the custody of one party, to that of another. Fiscal Custody Transfer measurements are required to determine the sales value for the producer, and royalties due to the owners. Custody Transfer measurements are also required for, transportation and storage through other parties' facilities, to establish ownership of the fluids, and charging tariffs for use of facilities.

Custody Transfer measurements are distinct from other measurements because they are; regulated by a contractual agreement, or a legal jurisdiction, with controls to prevent fraud, and specifications defining the standard of measurement. A typical uncertainty for Custody Transfer measurement of pipeline liquids, defined in OIML R-117, Class 0.3A [Ref. 9], is ±0.3%OMV (Of Measured Value) whilst a typical gas Custody Transfer measurement uncertainty, defined in OIML R-137-1, Class 0.5 [Ref. 10], is ±1.0%OMV. In practice the uncertainty standard used is dependant on the contractual terms and regulatory environment, however the uncertainty is generally of a similar magnitude, approaching the minimum practical levels achievable with field measurement and calibration.

Regulations call for a statement of uncertainty for each Custody Transfer meter station, which is traceable to references, to confirm the meter station complies with the standards. Uncertainty can be stated for a type of meter that has been independently certified or may be found from a detailed mathematical analysis. An analysis is often needed to ensure the meter station meets the standard at all operating conditions and flow rates. Analysis of uncertainty using RSS (Root Sum Square) with, partial derivative, or perturbation, sensitivity terms is preferred for meter station instrumentation and flow calculations. MCS (Monte Carlo Simulation) should be considered for complex methods, such as AGA8 gas compressibility, or COSTALD light hydrocarbon thermal and compressibility.

Pipelines transporting hydrocarbon products to markets have Custody Transfer metering at each entry and discharge location, in order to distribute sales revenues to each producer, in proportion to the quantity of their product entering the pipeline. This process known as "Allocation" comprises a calculation procedure that adversely impacts the uncertainty of the allocated products. This is dependant on the relative rate of products entering the pipeline, and in extreme cases a small producers' uncertainty exposure can be increased by an order of magnitude. MCS or RSS Perturbation uncertainty analysis methods are the most suitable for allocation due to the large number of terms in the allocation equations that can defy analysis by partial differentiation, even with specialised mathematical software. MCS is preferred as it also deals with the dependency between terms within the allocation equations.

3 Uncertainty Overview

The uncertainty methods described here are in accordance with the ISO/IEC Guide 98: 1995, 2nd Edition; *"Guide to the expression of uncertainty in measurement" (GUM).* [Ref. 16] which is the over-arching uncertainty standard adopted by ANSI, BSI and others; to which all uncertainty standards must comply. This includes ISO 5168: 2005(E), *"Measurement of fluid flow – Procedures for the evaluation of uncertainties"* , [Ref. 6] which is specific to flow measurement. Monte Carlo Simulation is now an accepted method of uncertainty analysis that is covered by a supplement to "The GUM" in ISO/IEC NP Guide 98: 1995/DSuppl 1.2, *"Propagation of distributions using a Monte Carlo method"*, [Ref. 8]. Much of the work discussed here is covered in more detail in papers by the author and others, listed in the references.

Uncertainty is defined as the interval within which 95% of the measurement values are expected fall for a given measurement, illustrated in Figure 1 below, for a Normal (Gaussian) distribution.

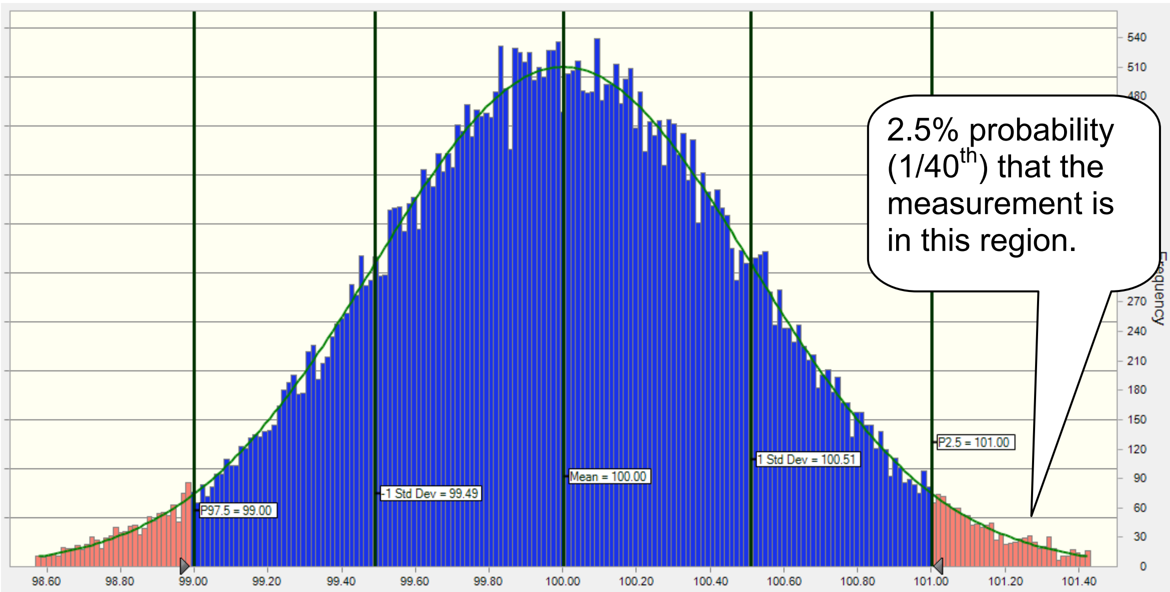


Figure 1 Uncertainty of ±1%OMV with a 95% confidence interval

For a normal distribution with a large number of random samples the 95% CI (Confidence Interval) is twice the standard deviation of the distribution. Put another way one in twenty readings will fall outside the uncertainty interval. For comparison, one standard deviation has a confidence interval of 68%, so one in three readings will fall outside the uncertainty interval.

Other confidence intervals may be used, including 90% and 99%, which by convention, and according to standards, should be stated along with the measurement uncertainty unless the confidence interval is 95%.

For a normal distribution the CI is found by multiplying the standard deviation by a CF (Coverage Factor) as follows:

CI	CF	
68%	1.00	One standard deviation
90%	1.65	
95%	1.96	Approximated to 2 by convention
99%	2.58	

These factors only apply where there are a large number of random samples. With population of five samples the CF for a CI of 95% increases to 2.57.

A typical measurement will have a Normal distribution however some instrument measurements have other characteristic distributions shown in Figure 2.

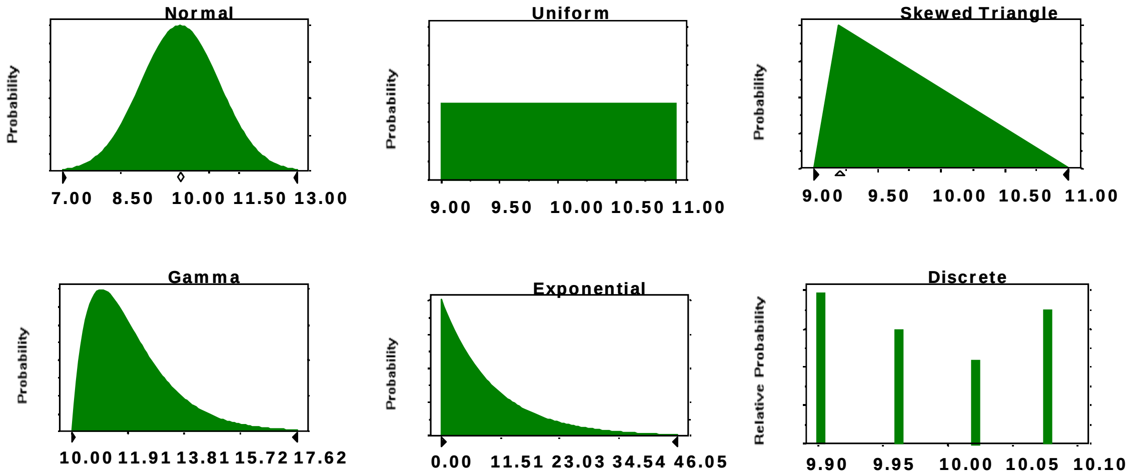
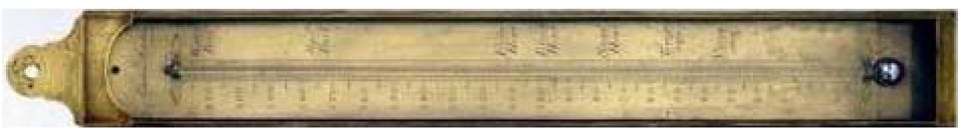


Figure 2 Measurement Interval Distributions

As an example a thermometer temperature scale has discrete steps, so temperature readings will be either of two values, with an equal probability that the temperature will fall on, or between, two points on the scale, and therefore will have a Uniform distribution. Uniform distributions are a common characteristic with a CF of √3 (1.72) at 95% CI.



A measurement will usually comprise several sources of uncertainty, so in the above example of a thermometer reading, there will also be a parallax reading error, with other sources of uncertainty such as the scale marking thickness, manufacturing tolerances, mercury meniscus etc.

The overall thermometer measurement uncertainty is found from the Root Sum Square (RSS) quadrature combination of uncertainties. This approach is based on the Central Limit Theorem (CLT) whereby combinations of uncertainty distributions will tend towards a Normal distribution, provided the sources of uncertainty are random.

Where the distribution of measurement is skewed, this will bias the mean and hence the uncertainty limits of a distribution, regardless of the apparent normality of the distribution. This can arise from a non-linear mathematical relationship between the measurement parameter and the resultant measurement.

The flow rate of a Venturi meter is proportional to the square root of the differential pressure measured between the bore and throat of the venturi. Figure 3 illustrates how the resultant distribution is skewed, thereby introducing bias in the resultant measurement uncertainty.

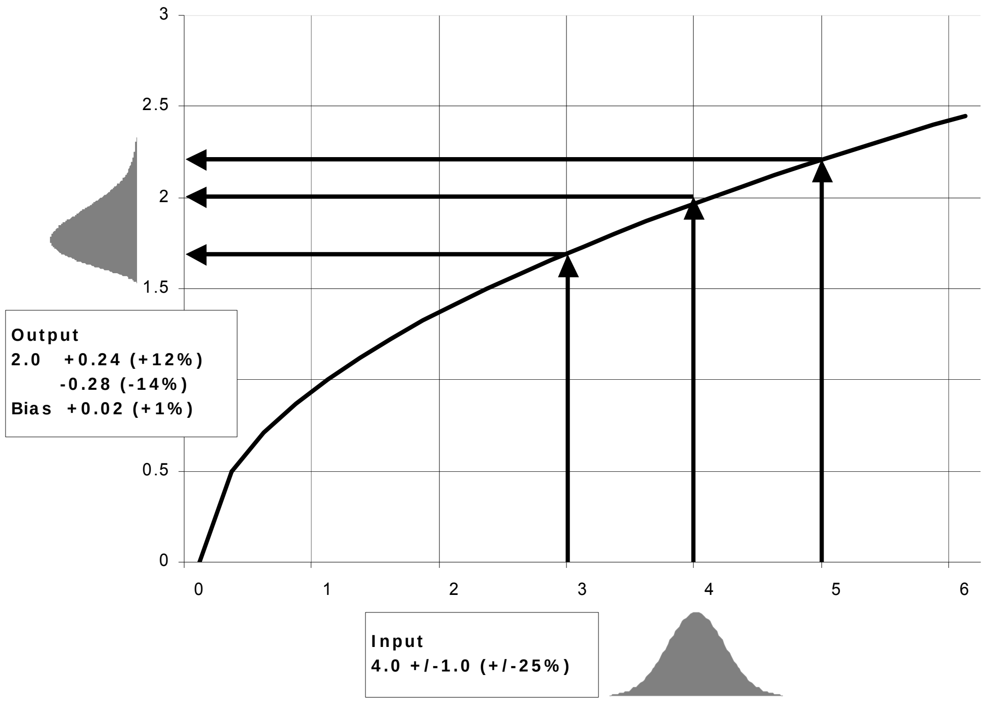


Figure 3 Distribution skew due to non-linearity

Where the bias is significant the skew due to the functional relationship can be calculated in order to offset mean value to the centre of the uncertainty limits before quadrature combination with other uncertainties.

4 Combination of Uncertainties

The combined uncertainty of a measurement can be found using the quadrature method described above or by simulating the uncertainty behavior of a measurement instrument with a technique known as Monte Carlo Simulation.

Using a model of the measurement the uncertainty of inputs to the model are simulated by generating several thousand samples reading with a normal, or other appropriate distribution. These simulation values propagate through the model generating a distribution for each output of the model. The standard deviation of each output distribution is found and from this the uncertainty with the advantage that the actual calculation is used rather than an abstract mathematical approximation, to ensure biases are apparent.

MCS and Quadrature uncertainty analysis methods are compared in a simple example to estimate the uncertainty of the length of a piece of string.

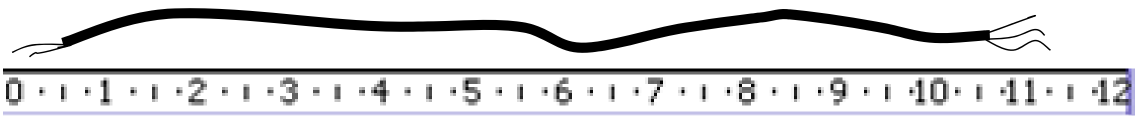


Figure 4 How long is a piece of string

The sources of uncertainty in Figure 4, How long is a piece of string, are estimated to be:

- String
  - Straightness 0.5” Normal distribution
  - Ends (not frays) 2 x 0.1” Normal distribution
  - Elasticity (stretch) 0.1” Normal distribution
  - Humidity 0.01” Normal distribution
- Ruler
  - Calibration 0.01” Normal distribution
  - Resolution (scale) 0.25” Rectangular distribution
  - Temperature 0.001” Normal distribution
- Reading
  - Parallax error 0.125” Rectangular distribution
  - Operator error (not included)

Quadrature

The sources of uncertainty are combined with the Quadrature method including division by a CF of 2 for normal, √3 for rectangular distributions.

$$\sqrt{\left(\frac{0.5}{2}\right)^2+2\left(\frac{0.1}{2}\right)^2+\left(\frac{0.1}{2}\right)^2+\left(\frac{0.01}{2}\right)^2+\left(\frac{0.01}{2}\right)^2+\left(\frac{0.25}{\sqrt{3}}\right)^2+\left(\frac{0.001}{2}\right)^2+\left(\frac{0.125}{\sqrt{3}}\right)^2}\cdot 2=0.62$$

10.25” +/- 0.62” or +/- 6.05% with 95% confidence level

Dominated by 0.5” string straightness

Monte Carlo Simulation

With MCS distributions are generated centered on zero, with a standard deviation of the magnitude of the uncertainty source, divided by the CF of 2 for normal distributions, and at the full magnitude of the uncertainty for the two rectangular distributions. The distribution trial values are summed for each set of trials and repeated several thousand times.

The standard deviation of the resultant distribution, shown in Figure 5, is found and adjusted to the 95% CI with a CF of 2 to find the combined uncertainty. This distribution confirms the CLT, that resulting Normal distribution is generated from the combination of Normal and Rectangular (Uniform) distributions, and therefore the coverage factor of √3 was correctly applied in the Quadrature solution.

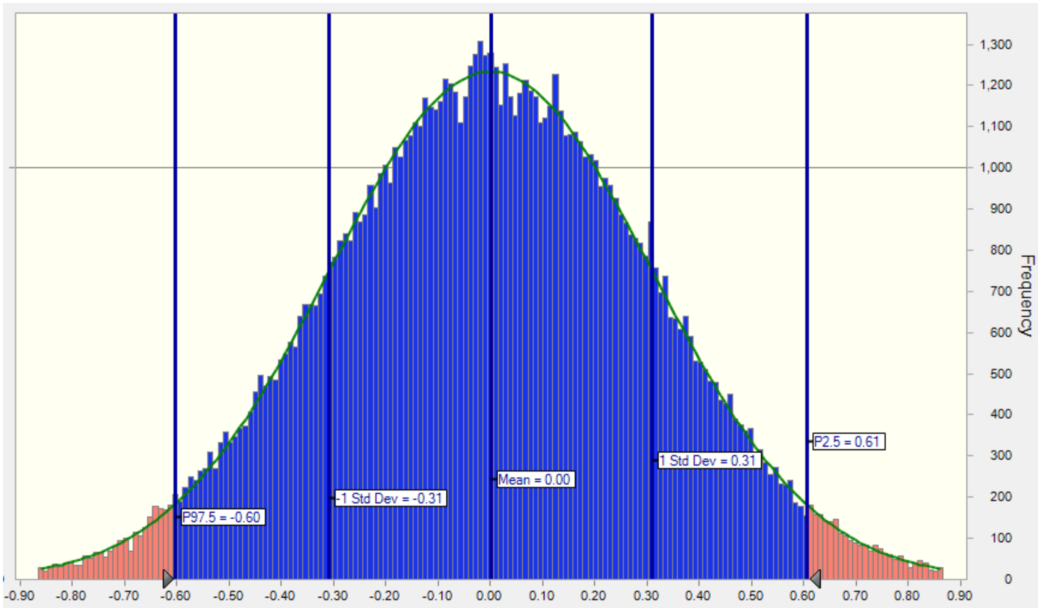


Figure 5 MCS combined uncertainty distribution

The results in Figure 6 comparing the combined uncertainty found by Quadrature and MCS, show that for a simple uncertainty analysis the results are the same. This has the advantage of providing a means to independently validate Quadrature uncertainty analysis results.

Observation	10.25"					
Description	Uncertainty	Sensitivity	Distribution	Coverage	(Unc/Cov)^2	MCS Trial Value
Straighthness	0.500"	1.00	Normal	2	0.06250"	0.00000"
End 1	0.100"	1.00	Normal	2	0.00250"	0.00000"
End 2	0.100"	1.00	Normal	2	0.00250"	0.00000"
Elasticity	0.100"	1.00	Normal	2	0.00250"	0.00000"
Humidity	0.010"	1.00	Normal	2	0.00003"	0.00000"
Calibration	0.010"	1.00	Normal	2	0.00003"	0.00000"
Resolution	0.250"	1.00	Rectangular	1.73	0.02088"	0.00000"
Temperature	0.001"	1.00	Normal	2	0.00000"	0.00000"
Parallax Error	0.125"	1.00	Rectangular	1.73	0.00522"	0.00000"
				2	0.09615"	0.00000"
				Method	RSS	MCS
				Uncertainty (")	0.62"	0.62''
				Uncertainty (%)	6.05%	6.05%

Figure 6 Comparison of combined uncertainty by Quadrature and MCS

5 Uncertainty Analysis

The uncertainty analysis in the previous example was applied to a single measurement parameter and therefore there was no requirement to consider the sensitivity of uncertainty sources, which in this case are all unity. In practice few measurements can be treated in such a simple fashion and the sensitivity of the result to each source of uncertainty must be considered.

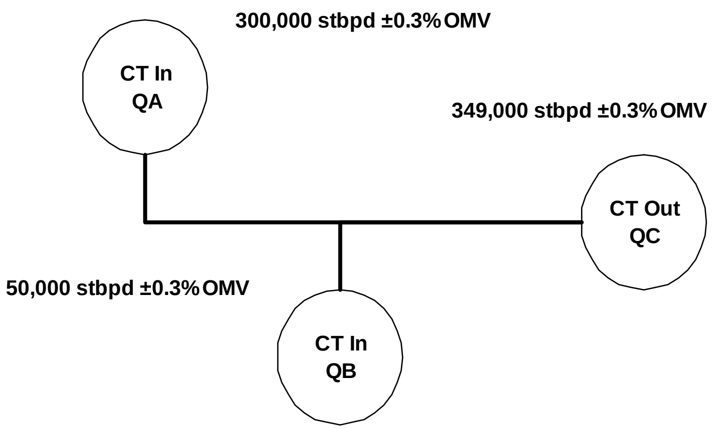


Figure 7 Pipeline allocation example

In the this example Quadrature analysis is compared with MCS analysis for a very simple pipeline allocation example with two entrants and a single discharge in Figure 7 showing the custody transfer meters, flow rates and uncertainty including a -0.3%OMV discrepancy between the sum of the two entrants and the discharge.

The allocation is found by distributing the quantity at the discharge meter proportionally to each pipeline entrant as follows:

$$AQ_A = \frac{Q_A \cdot Q_C}{Q_A + Q_B}$$ Allocation to entrant A

$$AQ_B = \frac{Q_B \cdot Q_C}{Q_A + Q_B}$$ Allocation to entrant B

For a given functional relationship:

$$y = f\left(X_1, X_2, ...X_N\right)$$

The uncertainty is found by Quadrature from the product of relative (percentage) measurement uncertainty (U), with measurement value (X) and the sensitivity (Θ) to the result as follows:

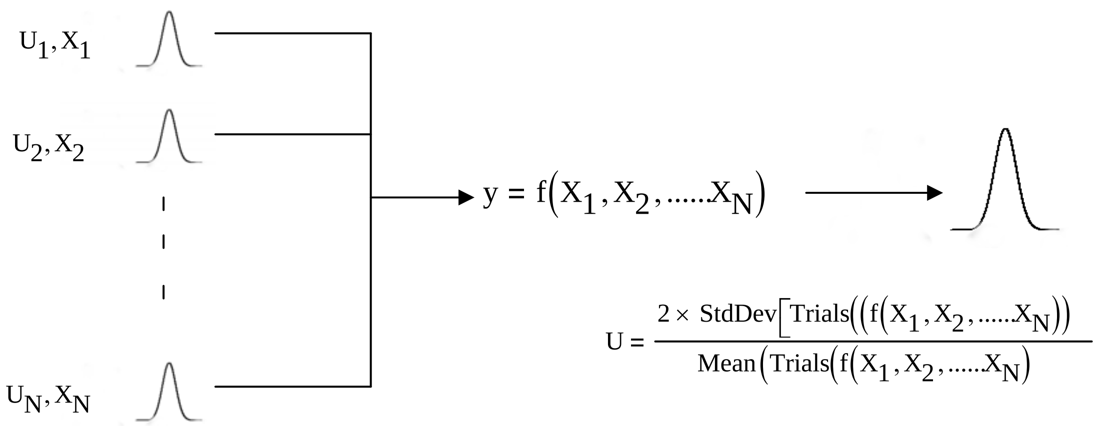
$$U = \frac{\sqrt{\left(\Theta_1 \cdot U_1 \cdot X_1\right)^2 + \left(\Theta_2 \cdot U_2 \cdot X_2\right)^2 + ... + \left(\Theta_N \cdot U_N \cdot X_N\right)^2}}{y}$$

To find the combined uncertainty of these allocated quantities by Quadrature the sensitivity (Θ) of the allocation (AQ) to each term in the respective allocation equation is found either from the partial derivatives or by perturbation of each term by the uncertainty. This second approach can be adapted to a numerical solution that does not explicitly require the sensitivity terms. The changes in the result due to a deviation (perturbation) of each term by the uncertainty of each term, are combined by Quadrature to find a combined uncertainty, as follows:

$$U = \frac{\sqrt{\begin{array}{l}\left[y - f\left[\left(X_1 - U_1 \cdot X_1\right), X_2, ...X_N\right]\right]^2 ... \\ + \left[y - f\left[X_1, \left(X_2 - U_2 \cdot X_2\right), ...X_N\right]\right]^2 ... \\ + ... \\ + \left[y - f\left[X_1, X_2, ...\left(X_N - U_N \cdot X_N\right)\right]\right]^2\end{array}}}{y}$$

A detailed analysis of the uncertainty found by partial derivatives and perturbation in Appendix A demonstrates that both methods produce the same result for this simple example.

Monte Carlo Simulation uncertainty analysis can be represented by a distribution of trial values for each input term with a mean of (X) and standard deviation of ((U x X)/2). The trials values are passed through the function (f()) and the uncertainty of the result is found from; twice the standard deviation of the trials, divided by the mean of the trials, as follows:



The MCS and both Quadrature combined allocation uncertainty analysis methods were implemented in a spreadsheet and with the same results shown in Figure 8 below confirming the suitability for this application.

Measurement Quantity and Uncertainty		
Stream	Flow rate	Uncertainty
Q_A	300,000	0.30%
Q_B	50,000	0.30%
Q_C	349,000	0.30%

Monte Carlo Simulation Uncertainty				
Allocation	Input Trial Qtmcs	Allocation Trials Aqtmcs	Allocation Mean AQmcs	Uncertainty UAQmcs
AQ_{Amcs}	-	#DIV/0!	299,143	0.31%
AQ_{Bmcs}	-	#DIV/0!	49,857	0.47%
Q_C	-			

Quadarature Partial Derivitive Uncertainty				
Allocation	Allocation AQpd	Sensitivity ΘAQ_{Apd}	Sensitivity ΘAQ_{Bpd}	Uncertainty UAQpd
AQ_{Apd}	299,143	0.14	- 0.14	0.31%
AQ_{Bpd}	49,857	- 0.85	0.85	0.47%
Q_C		0.86	0.14	

Quadrature Perturbation Uncertainty				
Allocation	Allocation AQpt	Deviation ΔAQ_{Apt}	Deviation ΔAQ_{Bpt}	Uncertainty UAQpt
AQ_{Apt}	299,143	128.53	- 128.53	0.31%
AQ_{Bpt}	49,857	- 128.26	128.26	0.47%
Q_C		897.43	149.57	

Figure 8 Comparison of MCS and Quadrature uncertainty results

The allocation procedure impacted the smaller producer, Entrant B, by over 50% additional uncertainty compared to the Custody Transfer meter. By comparison the increase in uncertainty due to allocation for the larger producer, Entrant A, was negligible.

The results in Figure 9 show the affect of increasing the entrants Custody Transfer meters uncertainty from ±0.3%OMV to ±1.0%OMV at the same rates as Figure 8. Whilst the Entrant B's uncertainty is increased by 25% and the disparity between both entrants has increased to 350%.

Measurement Quantity and Uncertainty		
Stream	Flow rate	Uncertainty
Q_A	300,000	1.00%
Q_B	50,000	1.00%
Q_C	349,000	0.30%

Monte Carlo Simulation Uncertainty				
Allocation	Input Trial Qtmcs	Allocation Trials Aqtmcs	Allocation Mean AQmcs	Uncertainty UAQmcs
AQ_{Amcs}	-	#DIV/0!	299,138	0.36%
AQ_{Bmcs}	-	#DIV/0!	49,859	1.25%

Figure 9 Pipeline entrant uncertainty of ±1.0%OMV

These two examples illustrate that in order to get a true picture of pipeline entrants' uncertainty exposure after allocation the uncertainty of the Custody Transfer meter and the relative rates of production for each of the pipeline entrants must be taken into account.

6 Applications

The previous sections have demonstrated that uncertainty obtained by Quadrature and MCS analysis methods are the same for simple cases. Quadrature for simple uncertainty analysis without sensitivity is easily implemented. This also applies with more complex calculations where sensitivity is required with two Quadrature variants and MCS available.

Some pipelines may have ten or more entrants and it becomes apparent that the number of sensitivity terms required increases by the square of the number of pipeline entrants. Partial derivative analysis becomes impractical due to the large number of terms which must be individually solved and cannot be automated. A practical maximum is around twelve entrants which also applies to perturbation Quadrature although this can be automated.

MCS uncertainty analysis only requires the allocation procedure in equation form which can be changed easily if another entrant joins the pipeline. On a gas pipeline Gas Sales are often allocated by molecular mass with up to twenty molecular components including liquids and can only be implemented with MCS. A recent project required a mass component allocation of products from three Custody Transfer meters to forty input streams with a total of around 1,200 inputs comprising mass flow rate, hydrocarbon compositions and density. This application had a high degree of dependency that is not taken into account with quadrature methods and had these been used the final allocation uncertainty would have been overstated compared to MCS.

Gas compressibility with the AGA8 detail characterization method is a good example of a complex calculation method that cannot be implemented using Quadrature. AGA8 utilizes two equations of state in a program to find the compressibility with a known gas composition at pressure and temperature. The AGA8 calculation also has an uncertainty that was estimated based on laboratory density methods and varies with pressure and temperature. An implementation in Figure 10 shows the inputs and uncertainty for the gas composition, pressure, temperature and the AGA8 method uncertainty for line and standard density.

AGA8 Gas Density				
Line Conditions	**Measurement**	**Uncertainty**		
Temperature deg C	25.56	0.450		
Pressure bara	18.05	0.755		
Gas Composition	**Compostion mol%**	**Normalised mol%**	**Component Uncertainty %**	**Uncertainty mol%**
Nitrogen mol%	0.720	0.720	1.00%	0.0072
Carbon Dioxide mol%	1.360	1.360	1.00%	0.0136
Methane mol%	85.330	85.330	2.00%	1.7066
Ethane mol%	6.150	6.150	1.00%	0.0615
Propane mol%	3.810	3.810	1.00%	0.0381
n-Butane mol%	2.020	2.020	1.00%	0.0202
i-Butane mol%	0.000	0.000	1.00%	0.0000
n-Pentane mol%	0.580	0.580	1.00%	0.0058
i-Pentane mol%	0.000	0.000	1.00%	0.0000
n-Hexane mol%	0.030	0.030	1.00%	0.0003
n-Heptane mol%	0.000	0.000	1.00%	0.0000
n-Octane mol%	0.000	0.000	0.00%	0.0000
n-Nonane mol%	0.000	0.000	0.00%	0.0000
n-Decane mol%	0.000	0.000	0.00%	0.0000
Total mol%	100.000	100.00		
Normalised	**True Result**	**Method Uncertainty**	**MCS Mean**	**MCS Uncertainty**
Line Density Kg/m3 (AGA8)	14.98	0.10%	14.99	4.42%
Standard Density Kg/m3 (AGA8)	0.8311	0.10%	0.83	0.34%
Line/Standard	18.03		18.03	4.41%

Figure 10 AGA8 detail characterization method MCS uncertainty

This approach can be extended to more complex meter stream uncertainty with a generic MCS model illustrated in Figure 11 that can be configured for any measurement configuration.

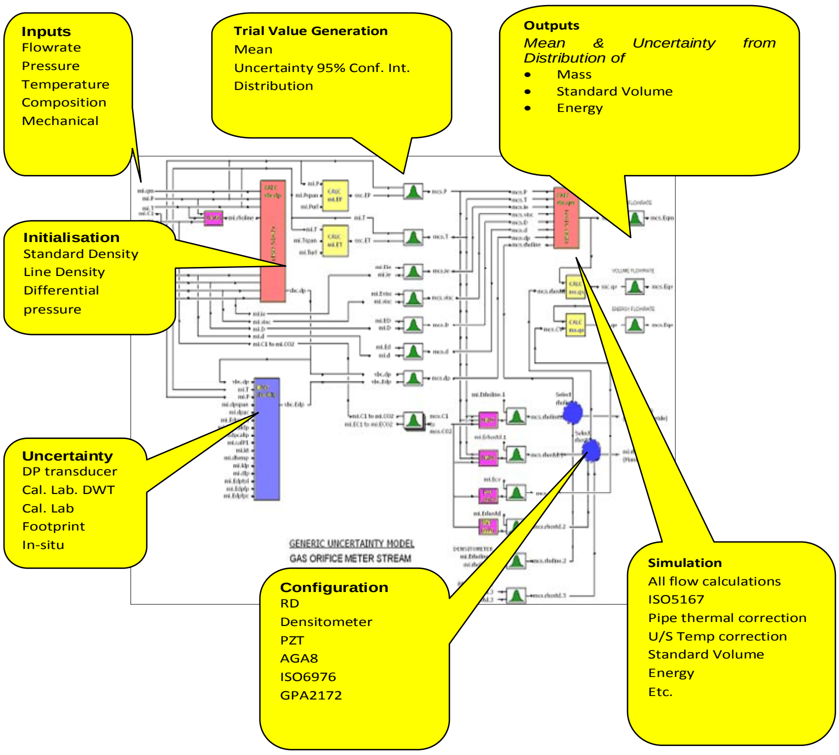


Figure 11 Gas Custody Transfer Meter

MCS trials values can be propagated to any level to build up a meter station which in turn becomes part of the allocation system shown in Figure 12

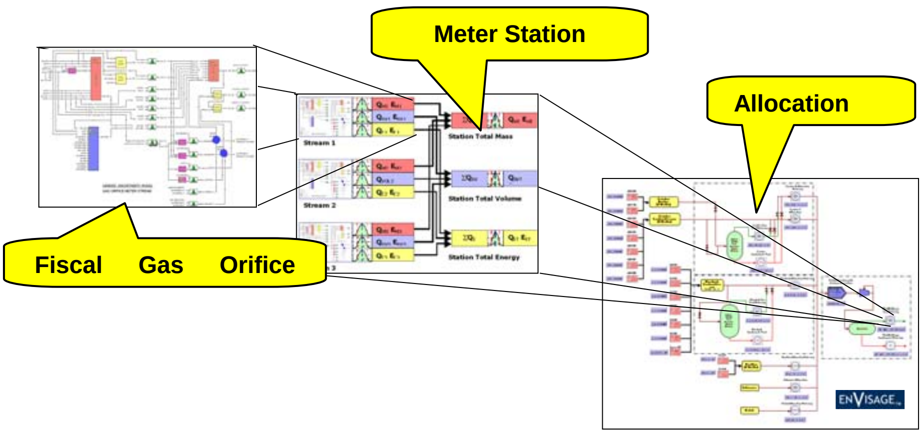


Figure 12 MCS System Uncertainty Model

7 Conclusions

1. The combined uncertainty found with the three uncertainty analysis methods gave the same results for simple cases;
2. Quadrature analysis is the most suitable for simple uncertainty cases including individual instruments;
3. Quadrature analysis with partial derivatives is impractical for expressions that have more than fifteen terms because the number of partial derivatives increases by the square of the number of terms;
4. Quadrature analysis using perturbation can be automated for cases with more than fifteen terms but is better suited to individual instruments and expressions with less than fewer terms;
5. Monte Carlo Simulation uses a different approach to uncertainty analysis and therefore can be utilised for independent validation of Quadrature methods;
6. A Custody Transfer meter uncertainty is not the uncertainty of the allocation which may be much greater particularly for small producers entering a pipeline with a large number of producers;
7. Dependency in expressions is automatically taken into account with Monte Carlo Simulation;
8. Monte Carlo Simulation distribution trial values can be propagated through every level from the basic measurements to complex mass component allocations systems.

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10 OIML R-137-1: 2006(E) *"Gas meters, Part 1: Requirements"*

Appendix A Quadrature Uncertainty Combination

Pipeline Measurement and Uncertainty

$Q_A := 300000$	$UQ_A := 0.3\%$	Entrant A
$Q_B := 50000$	$UQ_B := 0.3\%$	Entrant B
$Q_C := 349000$	$UQ_C := 0.3\%$	Discharge C

<u>Quadrature uncertainty combination with sensitivity by partial derivitive.</u>

Pipeline entrant A allocation

$$AQ_A := \frac{Q_A \cdot Q_C}{Q_A + Q_B} \qquad AQ_A = 299142.86$$

Entrant A allocation sensitivity terms found by partial differentiation

$$\Theta AQ_{AQA} := \frac{Q_B \cdot Q_C}{\left(Q_A + Q_B\right)^2} \qquad \Theta AQ_{AQA} = 0.14 \qquad \frac{\partial}{\partial Q_A} \frac{Q_A \cdot Q_C}{Q_A + Q_B} = 0.14$$

$$\Theta AQ_{AQB} := \frac{-Q_A \cdot Q_C}{\left(Q_A + Q_B\right)^2} \qquad \Theta AQ_{AQB} = -0.85 \qquad \frac{\partial}{\partial Q_B} \frac{Q_A \cdot Q_C}{Q_A + Q_B} = -0.85$$

$$\Theta AQ_{AQC} := \frac{Q_A}{Q_A + Q_B} \qquad \Theta AQ_{AQC} = 0.86 \qquad \frac{\partial}{\partial Q_C} \frac{Q_A \cdot Q_C}{Q_A + Q_B} = 0.86$$

Entrant A allocation uncertainty by Quadrature with partial derivitive senstivity terms

$$UAQ_A := \frac{\sqrt{\left(UQ_A \cdot Q_A \cdot \Theta AQ_{AQA}\right)^2 + \left(UQ_B \cdot Q_B \cdot \Theta AQ_{AQB}\right)^2 + \left(UQ_C \cdot Q_C \cdot \Theta AQ_{AQC}\right)^2}}{AQ_A}$$

$$UAQ_A = 0.31 \cdot \%$$

Pipeline entrant B allocation

$$AQ_B := \frac{Q_B \cdot Q_C}{Q_A + Q_B} \qquad AQ_B = 49857.14$$

Entrant A allocation sensitivity terms found by partial differentiation

$$\Theta AQ_{BQA} := \frac{-Q_B \cdot Q_C}{(Q_A + Q_B)^2} \qquad \Theta AQ_{BQA} = -0.14 \qquad \frac{\partial}{\partial Q_A} \frac{Q_B \cdot Q_C}{Q_A + Q_B} = -0.14$$

$$\Theta AQ_{BQB} := \frac{Q_A \cdot Q_C}{(Q_A + Q_B)^2} \qquad \Theta AQ_{BQB} = 0.85 \qquad \frac{\partial}{\partial Q_B} \frac{Q_B \cdot Q_C}{Q_A + Q_B} = 0.85$$

$$\Theta AQ_{BQC} := \frac{Q_B}{Q_A + Q_B} \qquad \Theta AQ_{BQC} = 0.14 \qquad \frac{\partial}{\partial Q_C} \frac{Q_B \cdot Q_C}{Q_A + Q_B} = 0.14$$

Entrant B allocation uncertainty by Quadrature with partial derivitive senstivity terms

$$UAQ_B := \frac{\sqrt{(UQ_A \cdot Q_A \cdot \Theta AQ_{BQA})^2 + (UQ_B \cdot Q_B \cdot \Theta AQ_{BQB})^2 + (UQ_C \cdot Q_C \cdot \Theta AQ_{BQC})^2}}{AQ_B}$$

$$UAQ_B = 0.47 \cdot \%$$

Quadrature uncertainty combination with sensitivity by perturbation.

Entrant A allocation uncertainty by Quadrature with perturbation senstivity terms

$$UAQ_A := \frac{\sqrt{\left[AQ_A - \frac{(Q_A - Q_A \cdot UQ_A) \cdot Q_C}{(Q_A - Q_A \cdot UQ_A) + Q_B}\right]^2 \ldots + \left[AQ_A - \frac{Q_A \cdot Q_C}{Q_A + (Q_B - Q_B \cdot UQ_B)}\right]^2 \ldots + \left[AQ_A - \frac{Q_A \cdot (Q_C - Q_C \cdot UQ_C)}{Q_A + Q_B}\right]^2}}{AQ_A}$$

$UAQ_A = 0.31 \cdot \%$

Entrant B allocation uncertainty by Quadrature with perturbation senstivity terms

$$UAQ_B := \frac{\sqrt{\left[AQ_B - \frac{Q_B \cdot Q_C}{(Q_A - Q_A \cdot UQ_A) + Q_B}\right]^2 \ldots + \left[AQ_B - \frac{(Q_B - Q_B \cdot UQ_B) \cdot Q_C}{Q_A + (Q_B - Q_B \cdot UQ_B)}\right]^2 \ldots + \left[AQ_B - \frac{Q_B \cdot (Q_C - Q_C \cdot UQ_C)}{Q_A + Q_B}\right]^2}}{AQ_B}$$

$UAQ_B = 0.47 \cdot \%$